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p-ADIC HAAR MULTIRESOLUTION ANALYSIS

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ABSTRACT. In this paper, the notion of *p-adic multiresolution analysis (MRA)* is introduced. We use a "natural" refinement equation whose solution (a refinable function) is the characteristic function of the unit disc. This equation reflects the fact that the characteristic function of the unit disc is the sum of p characteristic functions of disjoint discs of radius p^{-1}. The case $p = 2$ is studied in detail. Our MRA is a 2-adic analog of the real Haar MRA. But in contrast to the real setting, the refinable function generating our Haar MRA is periodic with period 1, which never holds for real refinable functions. This fact implies that there exist infinity many different 2-adic orthonormal wavelet bases in $\mathcal{L}^2(\mathbb{Q}_2)$ generated by the same Haar MRA. All of these bases are constructed. Since p-adic pseudo-differential operators are closely related to wavelet-type bases, our bases can be intensively used for applications.

1. INTRODUCTION

1.1. **p-Adic wavelets and pseudo-differential operators.** According to the well-known Ostrovsky theorem, *any nontrivial valuation on the field $\mathbb{Q}$ is equivalent either to the real valuation $|\cdot|$ or to one of the p-adic valuations* $|\cdot|_p$. We recall that the field $\mathbb{Q}_p$ of p-adic numbers is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: if an arbitrary rational number $x \neq 0$ is represented as $x = p^{\gamma}\frac{m}{n}$, where $\gamma = \gamma(x) \in \mathbb{Z}$, and m and n are not divisible by p, then

$$|x|_p = p^{-\gamma}, \quad x \neq 0, \qquad |0|_p = 0. \tag{1.1}$$

This norm in $\mathbb{Q}_p$ satisfies the strong triangle inequality $|x+y|_p \le \max(|x|_p, |y|_p)$.

Thus there are two equal in rights universes: the real universes and the p-adic one. The latter has a specific and unusual properties. Nevertheless, there are a lot of papers where different applications of p-adic analysis to physical problems, stochastics, cognitive sciences and psychology are studied [6]– [10], [13]– [19], [34]– [36] (see also the references therein). In view of the Ostrovsky theorem such investigations not only have great interest in

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itself, but lead to applications and better understanding of similar problems in *usual* mathematical physics.

We recall that there exists a p-adic analysis connected with the mapping $\mathbb{Q}_p$ into $\mathbb{Q}_p$ and an analysis connected with the mapping $\mathbb{Q}_p$ into the field of complex numbers $\mathbb{C}$, there exist two types of p-adic physics models. For the p-adic analysis related to the mapping $\mathbb{Q}_p \to \mathbb{C}$ the operation of partial differentiation is *not defined*, and as a result, large number of models connected with p-adic differential equations use pseudo-differential operators and the theory of p-adic distributions (generalized functions) (see the above mentioned papers and books). In particular, fractional operators D^α are extensively used in applications (see fore-quoted papers and especially [34]).

It is well known that the theory of p-adic pseudo-differential operators (in particular, fractional operators) and equations closely related to wavelet type bases. It is typical that p-adic compactly supported wavelets are eigenfunctions of p-adic pseudo-differential operators [3]– [5], [16], [17], [18], [20] – [22]. Thus the wavelet theory plays a key role in application of p-adic analysis and gives a new powerful technique for solving p-adic problems. This theory starts development only in resent years and has many open problems.

In [20], S. V. Kozyrev constructed the orthonormal compactly supported p-adic wavelet basis (1.2) in $\mathcal{L}^2(\mathbb{Q}_p)$:

$$\theta_{\gamma j a}(x) = p^{-\gamma/2}\chi_p\big(p^{-1}j(p^\gamma x - a)\big)\Omega\big(|p^\gamma x - a|_p\big), \quad x \in \mathbb{Q}_p, \tag{1.2}$$

$j \in J_p = \{1, 2, \dots, p-1\}$, $\gamma \in \mathbb{Z}$, $a \in I_p = \mathbb{Q}_p/\mathbb{Z}_p$. Kozyrev's wavelets (1.2) are eigenfunctions of the Vladimirov fractional operator [34, IX]. Further development and generalization of the theory of such type wavelets can be found in the papers by S. V. Kozyrev [21], [22], A. Yu. Khrennikov, and S. V. Kozyrev [16], [17], J. J. Benedetto, and R. L. Benedetto [8], and R. L. Benedetto [9]. In [3], the multidimensional p-adic wavelets generated by direct product of the Kozyrev one-dimensional wavelets were introduced. In [18], a new type of p-adic multidimensional wavelet basis was introduced:

$$\theta^{(m)}_{\gamma s a}(x) = p^{-\gamma/2}\chi_p\big(s(p^\gamma x - a)\big)\Omega\big(|p^\gamma x - a|_p\big), \quad x \in \mathbb{Q}_p,$$

where $s \in J_{p;m}$, $\gamma \in \mathbb{Z}$, $a \in I_p$. Here $J_{p;m} = \{s = p^{-m}\big(s_0 + s_1 p + \cdots + s_{m-1}p^{m-1}\big) : s_j = 0, 1, \dots, p-1; j = 0, 1, \dots, m-1; s_0 \neq 0\}$, $m \geq 1$ is a *fixed* positive integer. The multidimensional wavelets from [3] are a particular case of the last wavelets. Moreover, in [3], [18], there were derived the necessary and sufficient conditions for a class of multidimensional p-adic pseudo-differential operators (including fractional operator) to have such multidimensional wavelets as eigenfunctions.

It remains to point out that for pseudo-differential operators from [3], [18] a "natural" definition domain is the Lizorkin spaces of distributions $\Phi'(\mathbb{Q}_p^n)$, introduced in [3]. The space $\Phi'(\mathbb{Q}_p^n)$ is *invariant* under the mentioned above pseudo-differential operators. Moreover, the above mentioned p-adic wavelets belong to the Lizorkin space $\Phi(\mathbb{Q}_p^n)$ of test functions. Recall that the *usual*

Lizorkin spaces were studied in the excellent papers of P. I. Lizorkin [24], [25] (see also [29], [30]).

It's interesting to compare appearing first wavelets in p-adic analysis with the history of the wavelet theory in real analysis. In 1910 Haar [12] constructed an orthogonal basis for $\mathcal{L}_2(\mathbb{R})$ consisting of the dyadic shifts and scales of one piecewise constant function. A lot of mathematicians actively studied Haar basis, different kinds of generalizations were introduced, but during almost the whole century nobody could find another wavelet function (a function whose shifts and scales form an orthogonal basis). Only in early nineties a method for construction of wavelet functions appeared. This method is based on the notion of multiresolution analysis (MRA in the sequel) introduced by Y. Meyer and S. Mallat [28], [26], [27]. Smooth compactly supported wavelet functions were found in this way, which has been very important for some engineering applications. In this paper we introduce MRA in $\mathcal{L}_2(\mathbb{Q}_p)$ and present a concrete MRA for $p=2$ being an analog of Haar MRA in $\mathcal{L}_2(\mathbb{R})$. The same scheme as in the real setting leads to a Haar basis. It turned out that this Haar basis coincides with Kozyrev's wavelet system. However, 2-adic Haar MRA is not an identical copy of its real analog. In contrast to Haar MRA in $\mathcal{L}_2(\mathbb{R})$, we proved that there exist *infinity many different Haar orthogonal bases* in $\mathcal{L}_2(\mathbb{Q}_2)$ generated by the same MRA.

1.2. **Contents of the paper.** In Sec. 2, we recall some facts from the p-adic theory of distributions [11], [32], [33], [34]. In Sec. 3, some facts from the theory of the p-adic Lizorkin spaces [3] are recalled.

In Sec. 4, by Definition 4.1 we introduce the MRA adapted to the p-adic case. In Subsec. 4.2, we introduce the *refinement equation* (4.7)

$$\phi(x)=\sum_{r=0}^{p-1}\phi\Big(\frac{1}{p}x-\frac{r}{p}\Big), \quad x\in\mathbb{Q}_p,$$

whose solution $\phi(x)=\Omega\big(|x|_p\big)$ is the characteristic function of the unit disc, where where $\Omega(t)$ is the characteristic function of the interval $[0,1]$. The conjecture to use the above equation as the *refinement equation* was proposed in [18]. The above *refinement equation* is *natural* and reflects the fact that the characteristic function $\Omega\big(|x|_p\big)$ of the unit disc B_0 is represented as a sum of p pieces characteristic functions of the disjoint discs $B_{-1}(r)$, $r=0,1,\dots,p-1$ (see (2.7)).

In Subsec. 4.3, the 2-adic MRA is constructed. Namely, we proved that MRA is generated by a *refinable function* which is the characteristic function $\phi(x)=\Omega\big(|x|_2\big)$ of the unit disc $B_0=\{x: |x|_2\le 1\}\subset\mathbb{Q}_2$ and satisfies the *refinement equation* (4.8)

$$\phi(x)=\sum_{r=0}^{1}\phi\Big(\frac{1}{2}x-\frac{r}{2}\Big), \quad x\in\mathbb{Q}_2.$$

By our MRA we construct 2-adic orthonormal wavelet basis (4.15) in $\mathcal{L}^2(\mathbb{Q}_2)$, which is the Kozyrev basis (1.2) for the case $p=2$. It turned out that the Kozyrev wavelet basis is not unique orthonormal wavelet basis.

In Sec. 5, *infinity many different* 2*-adic wavelet orthonormal bases* in $\mathcal{L}^2(\mathbb{Q}_2)$ are constructed. Namely, using Theorem 5.1, we construct wavelet functions $\psi^{(s)}(x)$, $s\in\mathbb{N}$ whose dilatations and shifts form 2-adic orthonormal wavelet bases in $\mathcal{L}^2(\mathbb{Q}_2)$.

Since many p-adic models use pseudo-differential operators, in particular, fractional operator, these results on p-adic wavelets can be intensively used in applications. Moreover, p-adic wavelets can be used to construct solutions of linear and *semi-linear* pseudo-differential equations [5], [23].

2. p-Adic distributions

We recall some facts from the theory of p-adic distributions (generalized functions). Here and in what follows, we shall systematically use the notations and results from [34] and [11, Ch.II]. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ be the sets of positive integers, integers, complex numbers, respectively, and $\mathbb{N}_0=\{0\}\cup\mathbb{N}$. Denote by $\mathbb{Q}_p^*=\mathbb{Q}_p\setminus\{0\}$ the multiplicative group of the field $\mathbb{Q}_p$.

The canonical form of a p-adic number $x\neq 0$ is

$$x=p^{\gamma}(x_0+x_1p+x_2p^2+\cdots), \tag{2.1}$$

where $\gamma=\gamma(x)\in\mathbb{Z}$, $x_j=0,1,\dots,p-1$, $x_0\neq 0$, $j=0,1,\dots$. The series is convergent in the p-adic norm (1.1), and one has $|x|_p=p^{-\gamma}$. By means of representation (2.1), the *fractional part* $\{x\}_p$ of a number $x\in\mathbb{Q}_p$ is defined as follows

$$\{x\}_p=\begin{cases} 0, \quad \text{if} \quad \gamma(x)\ge 0 \quad \text{or} \quad x=0,\\ p^{\gamma}(x_0+x_1p+x_2p^2+\cdots+x_{|\gamma|-1}p^{|\gamma|-1}), \quad \text{if} \quad \gamma(x)<0. \end{cases} \tag{2.2}$$

The function

$$\chi_p(\xi x)=e^{2\pi i\{\xi x\}_p} \tag{2.3}$$

for every fixed $\xi\in\mathbb{Q}_p$ is an *additive character* of the field $\mathbb{Q}_p$.

According to [34, III.2.], any *multiplicative character* π of the field $\mathbb{Q}_p$ can be represented as

$$\pi(x)\stackrel{def}{=}\pi_\alpha(x)=|x|_p^{\alpha-1}\pi_1(x), \quad x\in\mathbb{Q}_p^*,$$

where $\pi(p)=p^{1-\alpha}$ and $\pi_1(x)$ is a *normed multiplicative character* such that $\pi_1(x)=\pi_1(|x|_px)$, $\pi_1(p)=\pi_1(1)=1$, $|\pi_1(x)|=1$. We denote $\pi_0=|x|_p^{-1}$.

The space $\mathbb{Q}_p^n=\mathbb{Q}_p\times\cdots\times\mathbb{Q}_p$ consists of points $x=(x_1,\dots,x_n)$, where $x_j\in\mathbb{Q}_p$, $j=1,2\dots,n$, $n\ge 2$. The p-adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p=\max_{1\le j\le n}|x_j|_p, \quad x\in\mathbb{Q}_p^n, \tag{2.4}$$

where $|x_j|_p$ id defined by (1.1).

Denote by $B^n_\gamma(a)=\{x\in\mathbb{Q}^n_p: |x-a|_p\le p^\gamma\}$ the ball of radius p^γ with the center at a point $a=(a_1,\dots,a_n)\in\mathbb{Q}^n_p$ and by $S^n_\gamma(a)=\{x\in\mathbb{Q}^n_p: |x-a|_p=p^\gamma\}=B^n_\gamma(a)\setminus B^n_{\gamma-1}(a)$ its boundary (sphere), $\gamma\in\mathbb{Z}$. For $a=0$ we set $B^n_\gamma(0)=B^n_\gamma$ and $S^n_\gamma(0)=S^n_\gamma$. For the case $n=1$ we will omit the upper index n. It is clear that

$$B^n_\gamma(a)=B_\gamma(a_1)\times\cdots\times B_\gamma(a_n), \tag{2.5}$$

where $B_\gamma(a_j)=\{x_j: |x_j-a_j|_p\le p^\gamma\}\subset\mathbb{Q}_p$ is a disc of radius p^γ with the center at a point $a_j\in\mathbb{Q}_p$, $j=1,2\dots,n$.

Any two balls in $\mathbb{Q}^n_p$ either are disjoint or one contains the other. Every point of the ball is its center.

According to [34, I.3,Examples 1,2.], the disc B_γ is represented by the sum of $p^{\gamma-\gamma'}$ *disjoint* discs $B_{\gamma'}(a)$, $\gamma'<\gamma$:

$$B_\gamma=B_{\gamma'}\cup\cup_a B_{\gamma'}(a), \tag{2.6}$$

where $a=0$ and $a=a_{-r}p^{-r}+a_{-r+1}p^{-r+1}+\cdots+a_{-\gamma'-1}p^{-\gamma'-1}$ are the centers of the discs $B_{\gamma'}(a)$, $r=\gamma,\gamma-1,\gamma-2,\dots,\gamma'+1$, $0\le a_j\le p-1$, $a_{-r}\ne 0$. In particular, the disc B_0 is represented by the sum of p *disjoint* discs

$$B_0=B_{-1}\cup\cup_{r=1}^{p-1}B_{-1}(r), \tag{2.7}$$

where $B_{-1}(r)=\{x\in S_0: x_0=r\}=r+p\mathbb{Z}_p$, $r=1,\dots,p-1$; $B_{-1}=\{|x|_p\le p^{-1}\}=p\mathbb{Z}_p$; and $S_0=\{|x|_p=1\}=\cup_{r=1}^{p-1}B_{-1}(r)$. Here all the discs are disjoint. We call coverings (2.6) and (2.7) the *canonical covering* of the discs B_0 and B_γ, respectively.

On $\mathbb{Q}_p$ there exists the Haar measure, i.e., a positive measure dx invariant under shifts, $d(x+a)=dx$, and normalized by the equality $\int_{|\xi|_p\le 1}dx=1$. The invariant measure dx on the field $\mathbb{Q}_p$ is extended to an invariant measure $d^nx=dx_1\cdots dx_n$ on $\mathbb{Q}^n_p$ in the standard way.

If f is an integrable function on $\mathbb{Q}_p$, then [11, Ch.II,§2.2], [34, IV]:

$$\begin{aligned}\int_{B_\gamma}dx &= p^\gamma,\\ \int_{B_N}f(x)\,dx &= \textstyle\sum_{\gamma=-\infty}^{N}\int_{S_\gamma}f(x)\,dx,\\ \int_{S_\gamma}f(x)\,dx &= \textstyle\int_{B_\gamma}f(x)\,dx-\int_{B_{\gamma-1}}f(x)\,dx.\end{aligned} \tag{2.8}$$

A complex-valued function f defined on $\mathbb{Q}^n_p$ is called *locally-constant* if for any $x\in\mathbb{Q}^n_p$ there exists an integer $l(x)\in\mathbb{Z}$ such that

$$f(x+y)=f(x),\quad y\in B^n_{l(x)}.$$

Let $\mathcal{E}(\mathbb{Q}^n_p)$ and $\mathcal{D}(\mathbb{Q}^n_p)$ be the linear spaces of locally-constant $\mathbb{C}$-valued functions on $\mathbb{Q}^n_p$ and locally-constant $\mathbb{C}$-valued functions with compact supports

(so-called test functions), respectively [34, VI.1.,2.]. If $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, according to Lemma 1 from [34, VI.1.], there exists $l \in \mathbb{Z}$, such that

$$\varphi(x+y) = \varphi(x), \quad y \in B_l^n, \quad x \in \mathbb{Q}_p^n.$$

The largest of such numbers $l = l(\varphi)$ is called the *parameter of constancy* of the function φ. Let us denote by $\mathcal{D}_N^l(\mathbb{Q}_p^n)$ the finite-dimensional space of test functions from $\mathcal{D}(\mathbb{Q}_p^n)$ having supports in the ball B_N^n and with parameters of constancy $\geq l$ [34, VI.2.]. The following embedding holds: $\mathcal{D}_N^l(\mathbb{Q}_p^n) \subset \mathcal{D}_{N'}^{l'}(\mathbb{Q}_p^n)$, $N \leq N'$, $l \geq l'$. Thus $\mathcal{D}(\mathbb{Q}_p^n) = \lim\operatorname{ind}_{N\to\infty} \lim\operatorname{ind}_{l\to-\infty} \mathcal{D}_N^l(\mathbb{Q}_p^n)$. The space $\mathcal{D}(\mathbb{Q}_p^n)$ is a complete locally convex vector space.

According to [34, VI,(5.2')], any function $\varphi \in \mathcal{D}_N^l(\mathbb{Q}_p^n)$ is represented in the following form

$$\varphi(x) = \sum_{\nu=1}^{p^{n(N-l)}} \varphi(c^\nu)\Delta_l(x - c^\nu), \quad x \in \mathbb{Q}_p^n, \tag{2.9}$$

where $\Delta_l(x - c^\nu)$ are the characteristic functions of the disjoint balls $B_l(c^\nu)$, and the points $c^\nu = (c_1^\nu, \dots c_n^\nu) \in B_N^n$ do not depend on φ.

Denote by $\mathcal{D}'(\mathbb{Q}_p^n)$ the set of all linear functionals on $\mathcal{D}(\mathbb{Q}_p^n)$ [34, VI.3.].

Let us introduce in $\mathcal{D}(\mathbb{Q}_p^n)$ a *canonical δ-sequence* $\delta_k(x) = p^{nk}\Omega(p^k|x|_p)$, and a *canonical 1-sequence* $\Delta_k(x) = \Omega(p^{-k}|x|_p)$, $k \in \mathbb{Z}$, $x \in \mathbb{Q}_p^n$, where

$$\Omega(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & t > 1. \end{cases} \tag{2.10}$$

Here $\Delta_k(x)$ is the characteristic function of the ball B_k^n. It is clear [34, VI.3., VII.1.] that $\delta_k \to \delta$, $k \to \infty$ in $\mathcal{D}'(\mathbb{Q}_p^n)$ and $\Delta_k \to 1$, $k \to \infty$ in $\mathcal{E}(\mathbb{Q}_p^n)$.

The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ is defined by the formula

$$F[\varphi](\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x)\varphi(x)\, d^n x, \quad \xi \in \mathbb{Q}_p^n,$$

where $\chi_p(\xi \cdot x) = \chi_p(\xi_1 x_1) \cdots \chi_p(\xi_n x_n) = e^{2\pi i \sum_{j=1}^n \{\xi_j x_j\}_p}$; $\xi \cdot x$ is the scalar product of vectors.

The Fourier transform is a linear isomorphism $\mathcal{D}(\mathbb{Q}_p^n)$ into $\mathcal{D}(\mathbb{Q}_p^n)$. Moreover, according to [32, Lemma A.], [33, III,(3.2)], [34, VII.2.],

$$\varphi(x) \in \mathcal{D}_N^l(\mathbb{Q}_p^n) \quad \text{iff} \quad F\big[\varphi(x)\big](\xi) \in \mathcal{D}_{-l}^{-N}(\mathbb{Q}_p^n). \tag{2.11}$$

We define the Fourier transform $F[f]$ of a distribution $f \in \mathcal{D}'(\mathbb{Q}_p^n)$ by the relation [34, VII.3.]:

$$\langle F[f], \varphi \rangle = \langle f, F[\varphi] \rangle, \quad \forall\, \varphi \in \mathcal{D}(\mathbb{Q}_p^n). \tag{2.12}$$

Let A be a matrix and $b \in \mathbb{Q}_p^n$. Then for a distribution $f \in \mathcal{D}'(\mathbb{Q}_p^n)$ the following relation holds [34, VII,(3.3)]:

$$F[f(Ax+b)](\xi) = |\det A|_p^{-1} \chi_p\big(-A^{-1}b \cdot \xi\big) F[f(x)]\big(A^{-1}\xi\big), \tag{2.13}$$

where $\det A \neq 0$. According to [34, IV,(3.1)],

$$F[\Delta_k](x) = \delta_k(x), \quad k \in \mathbb{Z}, \qquad x \in \mathbb{Q}_p^n. \tag{2.14}$$

In particular, $F[\Omega(|\xi|_p)](x) = \Omega(|x|_p)$.

The convolution $f * g$ for distributions $f, g \in \mathcal{D}'(\mathbb{Q}_p^n)$ is defined (see [34, VII.1.]) as

$$\langle f * g, \varphi \rangle = \lim_{k \to \infty} \langle f(x) \times g(y), \Delta_k(x)\varphi(x+y) \rangle \tag{2.15}$$

if the limit exists for all $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, where $f(x) \times g(y)$ is the direct product of distributions. If for distributions $f, g \in \mathcal{D}'(\mathbb{Q}_p^n)$ the convolution $f * g$ exists then [34, VII,(5.4)]

$$F[f * g] = F[f]F[g]. \tag{2.16}$$

Definition 2.1. Let π_α be a multiplicative character of the field $\mathbb{Q}_p$. A distribution $f \in \mathcal{D}'(\mathbb{Q}_p^n)$ is called *homogeneous* of degree π_α if for all $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ and $t \in \mathbb{Q}_p^*$ we have the relation

$$\Big\langle f, \varphi\Big(\frac{x_1}{t}, \dots, \frac{x_n}{t}\Big)\Big\rangle = \pi_\alpha(t)|t|_p^n \big\langle f, \varphi(x_1, \dots, x_n)\big\rangle$$

i.e., $f(tx) = f(tx_1, \dots, tx_n) = \pi_\alpha(t)f(x)$, $x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$. A *homogeneous* distribution of degree $\pi_\alpha(t) = |t|_p^{\alpha-1}$ ($\alpha \neq 0$) is called homogeneous of degree $\alpha - 1$.

3. The p-adic Lizorkin spaces

Let us introduce the p-adic *Lizorkin space of test functions*

$$\Phi(\mathbb{Q}_p^n) = \{\phi : \phi = F[\psi], \, \psi \in \Psi(\mathbb{Q}_p^n)\},$$

where

$$\Psi(\mathbb{Q}_p^n) = \{\psi(\xi) \in \mathcal{D}(\mathbb{Q}_p^n) : \psi(0) = 0\}.$$

Here $\Psi(\mathbb{Q}_p^n), \Phi(\mathbb{Q}_p^n) \subset \mathcal{D}(\mathbb{Q}_p^n)$. The space $\Phi(\mathbb{Q}_p^n)$ is called the p-adic *Lizorkin space of test functions*. The space $\Phi(\mathbb{Q}_p^n)$ can be equipped with the topology of the space $\mathcal{D}(\mathbb{Q}_p^n)$ which makes Φ a complete space.

In view of (2.11), the following lemma holds.

Lemma 3.1. ([3], [4]) (a) *$\phi \in \Phi(\mathbb{Q}_p^n)$ iff $\phi \in \mathcal{D}(\mathbb{Q}_p^n)$ and*

$$\int_{\mathbb{Q}_p^n} \phi(x)\, d^n x = 0. \tag{3.1}$$

(b) *$\phi \in \mathcal{D}_N^l(\mathbb{Q}_p^n) \cap \Phi(\mathbb{Q}_p^n)$, i.e., $\int_{B_N^n} \phi(x)\, d^n x = 0$, iff $\psi = F^{-1}[\phi] \in \mathcal{D}_{-l}^{-N}(\mathbb{Q}_p^n) \cap \Psi(\mathbb{Q}_p^n)$, i.e., $\psi(\xi) = 0$, $\xi \in B_{-N}^n$.*

Unlike the classical Lizorkin space, any function $\psi(\xi) \in \Phi(\mathbb{Q}_p^n)$ is equal to zero not only at $\xi = 0$ but in a ball $B^n \ni 0$, as well.

Let $\Phi'(\mathbb{Q}_p^n)$ denote the topological dual of the space $\Phi(\mathbb{Q}_p^n)$. We call it the p-adic *Lizorkin space of distributions*.

By $\Psi^{\perp}$ and $\Phi^{\perp}$ we denote the subspaces of functionals in $\mathcal{D}'(\mathbb{Q}_p^n)$ orthogonal to $\Psi(\mathbb{Q}_p^n)$ and $\Phi(\mathbb{Q}_p^n)$, respectively. Thus $\Psi^{\perp} = \{f \in \mathcal{D}'(\mathbb{Q}_p^n) : f = C\delta,\, C \in \mathbb{C}\}$ and $\Phi^{\perp} = \{f \in \mathcal{D}'(\mathbb{Q}_p^n) : f = C,\, C \in \mathbb{C}\}$.

Proposition 3.1. ([3])

$$\Phi'(\mathbb{Q}_p^n) = \mathcal{D}'(\mathbb{Q}_p^n)/\Phi^{\perp}, \qquad \Psi'(\mathbb{Q}_p^n) = \mathcal{D}'(\mathbb{Q}_p^n)/\Psi^{\perp}.$$

The space $\Phi'(\mathbb{Q}_p^n)$ can be obtained from $\mathcal{D}'(\mathbb{Q}_p^n)$ by "sifting out" constants. Thus two distributions in $\mathcal{D}'(\mathbb{Q}_p^n)$ differing by a constant are indistinguishable as elements of $\Phi'(\mathbb{Q}_p^n)$.

Similarly to (2.12), we define the Fourier transform of distributions $f \in \Phi'_{\times}(\mathbb{Q}_p^n)$ and $g \in \Psi'_{\times}(\mathbb{Q}_p^n)$ by the relations:

$$\begin{aligned} \langle F[f], \psi \rangle &= \langle f, F[\psi] \rangle, \qquad \forall\, \psi \in \Psi(\mathbb{Q}_p^n), \\ \langle F[g], \phi \rangle &= \langle g, F[\phi] \rangle, \qquad \forall\, \phi \in \Phi(\mathbb{Q}_p^n). \end{aligned} \tag{3.2}$$

By definition, $F[\Phi(\mathbb{Q}_p^n)] = \Psi(\mathbb{Q}_p^n)$ and $F[\Psi(\mathbb{Q}_p^n)] = \Phi(\mathbb{Q}_p^n)$, i.e., (3.2) give well defined objects.

4. Construction of multiresolution analysis

4.1. **p-Adic multiresolution analysis.** Denote the factor group $\mathbb{Q}_p/\mathbb{Z}_p$ by I_p, i.e.

$$\begin{aligned} I_p = \{a = p^{-\gamma}\big(a_0 + a_1 p + \cdots + a_{\gamma-1}p^{\gamma-1}\big) : \\ \gamma \in \mathbb{N};\, a_j = 0, 1, \dots, p-1;\, j = 0, 1, \dots, \gamma - 1\}. \end{aligned} \tag{4.1}$$

It is well known that $\mathbb{Q}_p = B_0 \cup \cup_{\gamma=1}^{\infty} S_\gamma$, where $S_\gamma = \{x \in \mathbb{Q}_p : |x|_p = p^{\gamma}\}$. In view of (2.1), $x \in S_\gamma$, $\gamma \ge 1$ if and only if $x = x_{-\gamma}p^{-\gamma} + x_{-\gamma+1}p^{-\gamma+1} + \cdots + x_{-1}p^{-1} + \xi$, where $\xi \in B_0$. Since $x_{-\gamma}p^{-\gamma} + x_{-\gamma+1}p^{-\gamma+1} + \cdots + x_{-1}p^{-1} \in I_p$, we have a "natural" decomposition of $\mathbb{Q}_p$ to a union of mutually disjoint discs:

$$\mathbb{Q}_p = \cup_{a \in I_p} B_0(a).$$

So, I_p is a "natural" group of shifts for $\mathbb{Q}_p$.

Definition 4.1. A collection of closed spaces $V_j \subset \mathcal{L}^2(\mathbb{Q}_p)$, $j \in \mathbb{Z}$ is called a *multiresolution analysis* (*MRA*) *in* $\mathcal{L}^2(\mathbb{Q}_p)$ if the following axioms hold

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\cup_{j\in\mathbb{Z}} V_j$ is dense in $\mathcal{L}^2(\mathbb{Q}_p)$;

(c) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $f(\cdot) \in V_j \iff f(p^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) there a function $\phi \in V_0$ such that the system $\phi(x-a)$, $a \in I_p$, form an orthonormal basis for V_0.

The function ϕ from axiom (e) is called *scaling* or *refinable*. It follows immediately from axioms (d) and (e) that the functions $p^{j/2}\phi(p^{-j}\cdot - a)$, $a \in I_p$, form an orthonormal basis for V_j.

According to the standard scheme (see, e.g., [31, §1.3]) for construction of MRA-based wavelets, for each j, we define a space W_j (*wavelet space*) as the orthogonal complement of V_j in V_{j+1}, i.e.,

$$V_{j+1} = V_j \oplus W_j, \qquad j \in \mathbb{Z}, \tag{4.2}$$

where $W_j \perp V_j$, $j \in \mathbb{Z}$. It is not difficult to see that

$$f \in W_j \Longleftrightarrow f(p^{-1}\cdot) \in W_{j+1}, \quad \text{for all} \quad j \in \mathbb{Z} \tag{4.3}$$

and $W_j \perp W_k$, $j \neq k$. Taking into account axioms (b) and (c), we obtain

$$\oplus_{j\in\mathbb{Z}} W_j = \mathcal{L}^2(\mathbb{Q}_p) \quad \text{(orthogonal direct sum)}. \tag{4.4}$$

If now we find a function $\psi \in W_0$ such that the system $\psi(x-a)$, $a \in I_p$, form an orthonormal basis for W_0, then the system $p^{j/2}\psi(p^{-j}\cdot -a)$, $a \in I_p$, is an orthonormal basis for $\mathcal{L}^2(\mathbb{Q}_p)$. Such a function ψ is called a *wavelet function* and the basis is a *wavelet basis*.

4.2. **p-Adic refinement equation.** Let ϕ be a refinable function for a MRA. As was mentioned above, the system $p^{1/2}\phi(p^{-1}\cdot -a)$, $a \in I_p$, is a basis for V_1. It follows from axoim (a) that

$$\phi = \sum_{a\in I_p} \alpha_a \phi(p^{-1}\cdot -a), \quad \alpha_a \in \mathbb{C}. \tag{4.5}$$

We see that the function ϕ is a solution of a special kind of functional equation. Such equations are called refinement equations. Investigation of refinement equations and their solutions is the most difficult part of wavelet theory in real analysis.

A natural way for construction of a MRA (see, e.g., [31, §1.2]) is the following. We start with an appropriate function ϕ whose integer shifts form an orthonormal system, and set $V_0 = \overline{\operatorname{span}\big\{\phi\big(x-a\big) : a \in I_p\big\}}$ and $V_j = \overline{\operatorname{span}\big\{\phi\big(p^{-j}x-a\big) : a \in I_p\big\}}$, $j \in \mathbb{Z}$. It is clear that axioms (d) and (e) of Definition 4.1 are fulfilled.

Of course, not any such a function ϕ provides axiom (a). In the *real setting*, the relation $V_0 \subset V_1$ holds if and only if the refinable function satisfies a refinement equation. Situation is different in p-adics. Generally speaking, a refinement equation (4.5) does not imply the including property $V_0 \subset V_1$. Indeed, we need all the functions $\phi(\cdot - b)$, $b \in I_p$, to belong to the space V_1, i.e., the equalities $\phi(x-b) = \sum_{a\in I_p} \alpha_{a,b}\phi(p^{-1}x-a)$ should be fulfilled for all $b \in I_p$. Since $p^{-1}b + a$ is not in I_p in general, we can not state that refinement equation (4.5) implies $\phi(x-b) = \sum_{a\in I_p} \alpha_{a,b}\phi(p^{-1}x - p^{-1}b - a) \in V_1$ for all $b \in I_p$.

The *refinement equation* reflects *some "self-similarity"*. The structure of the space $\mathbb{Q}_p$ has a *natural* "self-similarity" property which is given by formulas (2.6), (2.7). By (2.7), the characteristic function $\Delta_0(x) = \Omega\big(|x|_p\big)$ of the unit

disc B_0 is represented as a sum of p characteristic functions of the disjoint discs $B_{-1}(r)$, $r = 0, 1, \dots, p-1$, i.e.,

$$\Delta_0(x) = \sum_{r=0}^{p-1} \Delta_0\Big(\frac{1}{p}x - \frac{r}{p}\Big), \quad x \in \mathbb{Q}_p. \tag{4.6}$$

Thus, in p-adics, we have a *natural refinement equation* (4.5):

$$\phi(x) = \sum_{r=0}^{p-1} \phi\Big(\frac{1}{p}x - \frac{r}{p}\Big), \quad x \in \mathbb{Q}_p, \tag{4.7}$$

whose solution is $\phi(x) = \Delta_0(x) = \Omega\big(|x|_p\big)$. This equation is an analog of the *refinement equation* generating Haar MRA in real analysis.

4.3. **Construction of 2-adic Haar multiresolution analysis.** Now, using the *refinement equation* (4.7) for $p = 2$

$$\phi(x) = \phi\Big(\frac{1}{2}x\Big) + \phi\Big(\frac{1}{2}x - \frac{1}{2}\Big), \quad x \in \mathbb{Q}_2, \tag{4.8}$$

and its solution, the *refinable function* $\phi(x) = \Delta_0(x) = \Omega\big(|x|_2\big)$, we construct 2-adic multiresolution analysis.

Set

$$V_0 = \overline{\operatorname{span}\big\{\phi\big(x-a\big) : a \in I_2\big\}}, \tag{4.9}$$

$$V_j = \overline{\operatorname{span}\big\{\phi\big(2^{-j}x-a\big) : a \in I_2\big\}}, \quad j \in \mathbb{Z}. \tag{4.10}$$

It is clear that axioms (d) and (e) of Definition 4.1 are fulfilled and the system $2^{j/2}\phi(2^{-j}\cdot -a)$, $a \in I_p$ is an orthonormal basis for V_j, $j \in \mathbb{Z}$.

Note that the characteristic function of the unit disc $\Omega\big(|x|_2\big)$ has a wonderful feature: $\Omega(|\cdot+\xi|_2) = \Omega(|\cdot|_2)$, for all $\xi \in \mathbb{Z}_2$ because the p-adic norm is non-Archimedean. In particular, $\Omega(|\cdot\pm 1|_2) = \Omega(|\cdot|_2)$, i.e.,

$$\phi(x \pm 1) = \phi(x), \quad \forall\, x \in \mathbb{Q}_2. \tag{4.11}$$

Thus ϕ is *periodic* with the period 1.

In view of this fact, taking into account that $2^{-1}b + a \pmod 1$ is in I_2, for all $a, b \in I_2$, it follows from the refinement equation (4.8) that $V_0 \subset V_1$. By (4.10), this yields axiom (a).

Due to the *refinement equation* (4.8), we obtain that $V_j \subset V_{j+1}$, i.e., the axiom (a) from Definition 4.1 holds.

Lemma 4.1. *The axiom* (b) *of Definition* 4.1 *holds, i.e.,* $\overline{\cup_{j\in\mathbb{Z}}V_j} = \mathcal{L}^2(\mathbb{Q}_2)$.

Proof. According to (2.9), any function $\varphi \in \mathcal{D}(\mathbb{Q}_2)$ belongs to one of the spaces $\mathcal{D}^l_N(\mathbb{Q}_2)$, and consequently, is represented in the form

$$\varphi(x) = \sum_{\nu=1}^{p^{N-l}} \varphi(c^\nu)\Delta_l(x - c^\nu), \quad x \in \mathbb{Q}_2, \tag{4.12}$$

where $\Delta_l(\cdot - c^\nu)$ are the characteristic functions of the mutually disjoint discs $B_l(c^\nu) \subset \mathbb{Q}_2$, $c^\nu \in B_N$, $\nu = 1, 2, \dots p^{N-l}$; $l = l(\varphi)$, $N = N(\varphi)$. Since $\Delta_l(x - c^\nu) = \Omega(p^{-l}|x - c^\nu|_p) = \Omega(|p^l x - p^l c^\nu|_p)$ and any number $p^l c^\nu$ can be represented in the form $p^l c^\nu = a^\nu + b^\nu$, where $a^\nu \in I_2$, $b^\nu \in \mathbb{Z}_2$, we have $\Delta_l(x - c^\nu) = \Delta_l(x - a^\nu)$. Thus any function $\varphi \in \mathcal{D}(\mathbb{Q}_2)$ can be represented in the form

$$\varphi(x) = \sum_{\nu=1}^{p^{N-l}} \alpha_\nu \Delta_l(x - a^\nu), \quad x \in \mathbb{Q}_2, \quad a^\nu \in I_2, \quad \alpha_\nu \in \mathbb{C}. \tag{4.13}$$

Consequently, on the basis of (4.10), $\varphi(x) \in V_{-l}$. Thus any test function φ belongs to one of the space V_j, where $j = j(\varphi)$.

Since the space $\mathcal{D}(\mathbb{Q}_2)$ is dense in $\mathcal{L}^2(\mathbb{Q}_2)$ [34, VI.2], approximating any function from $\mathcal{L}^2(\mathbb{Q}_2)$ by test functions (4.13), we prove our assertion. $\Box$

Lemma 4.2. *The axiom* (*c*) *of Definition* 4.1 *holds, i.e.,* $\cap_{j\in\mathbb{Z}} V_j = \{0\}$.

Proof. Suppose that $\cap_{j\in\mathbb{Z}} V_j \neq \{0\}$. Then there exists a function $f \in V_j$ for all $j \in \mathbb{Z}$. Hence, due to (4.10), $f(x) = \sum_{a\in I_2} c_{ja}\phi\big(2^{-j}x - a\big)$ for all $j \in \mathbb{Z}$.

Let $x = 2^{-N}(x_0 + x_1 2 + x_2 2^2 + \cdots)$. Since $2^{-j}x = 2^{-N-j}(x_0 + x_1 2 + x_2 2^2 + \cdots)$, for all $j \le -N$, we have $2^{-j}x \in \mathbb{Z}_2$, and, consequently, $|2^{-j}x - a|_2 > 1$ for all $a \in I_2$, $a \neq 0$. Thus $\phi\big(2^{-j}x - a\big) = 0$ for all $j \le -N$ and $a \in I_2$, $a \neq 0$. Since $|2^{-j}x|_2 \le 1$, we have $f(x) = c_{j0}$ for all $j \le -N$. Similarly, for another $x' = 2^{-N'}(x'_0 + x'_1 2 + x'_2 2^2 + \cdots)$, we have $f(x') = c_{j'0}$ for all $j \le -N'$. This yields that $f(x) = f(x')$. Consequently, $f(x) \equiv C$, where C is a constant. However, if $C \neq 0$, $f \notin \mathcal{L}^2(\mathbb{Q}_2)$. Thus, $C = 0$ and the proof of the theorem is complete. $\Box$

According to the above scheme, we introduce the space W_0 as the orthogonal complement of V_0 in V_1.

Set

$$\psi^{(0)}(x) = \phi\Big(\frac{1}{2}x\Big) - \phi\Big(\frac{1}{2}x - \frac{1}{2}\Big). \tag{4.14}$$

Lemma 4.3. *The shift system* $\psi^{(0)}(x - a)$, $a \in I_2$, *is an orthonormal basis of the space* W_0.

Proof. Let us prove that $W_0 \perp V_0$. It follows from (4.8), (4.14) that

$$\big(\psi^{(0)}(x - a), \phi(x - b)\big) = \int_{\mathbb{Q}_2} \psi^{(0)}(x - a)\phi(x - b)\, dx$$

$$= \int_{\mathbb{Q}_2} \Big(\phi\Big(\frac{x}{2} - \frac{a}{2}\Big) - \phi\Big(\frac{x}{2} - \frac{1}{2} - \frac{a}{2}\Big)\Big)\Big(\phi\Big(\frac{x}{2} - \frac{b}{2}\Big) + \phi\Big(\frac{x}{2} - \frac{1}{2} - \frac{b}{2}\Big)\Big)\, dx$$

for all $a, b \in I_2$. Let $a \neq b$. Since it is impossible $a \neq b + 1$, $b \neq a + 1$, taking into account that the functions $2^{1/2}\phi(2^{-1}\cdot - c)$, $c \in I_2$ are orthonormal, we obtain $\big(\psi^{(0)}(x - a), \phi(x - b)\big) = 0$. If $a = b$, again due to the orthonormality

of the system $2^{1/2}\phi(2^{-1}\cdot -c)$, $c\in I_2$, taking into account that $\frac{a}{2}, \frac{a}{2}+\frac{1}{2}\in I_2$, we have

$$\big(\psi^{(0)}(x-a),\phi(x-a)\big)=\int_{\mathbb{Q}_2}\left(\phi^2\Big(\frac{x}{2}-\frac{a}{2}\Big)-\phi^2\Big(\frac{x}{2}-\frac{1}{2}-\frac{a}{2}\Big)\right)dx$$
$$=\int_{\mathbb{Q}_2}\phi\Big(\frac{x}{2}-\frac{a}{2}\Big)\,dx-\int_{\mathbb{Q}_2}\phi\Big(\frac{x}{2}-\frac{1}{2}-\frac{a}{2}\Big)\,dx=0.$$

Thus, $\psi^{(0)}(x+a)\perp\phi(x+b)$ for all $a,b\in I_2$.

The refinement equation (4.8) and relation (4.14) imply that

$$\phi\Big(\frac{x}{2}-a\Big)=\frac{1}{2}\Big(\phi(x-2a)+\psi^{(0)}(x-2a)\Big),\quad a\in I_2.$$

Since $\{2^{1/2}\phi(2^{-1}x-a): a\in I_2\}$ is a basis for V_1, we have $V_1=V_0\oplus W_0$, i.e., (4.2) holds. □

Thus we prove that the collection $\{V_j : j\in\mathbb{Z}\}$ is a MRA in $\mathcal{L}^2(\mathbb{Q}_2)$ and the function $\psi^{(0)}$ defined by (4.14) is a wavelet function. This MRA is a 2-adic analog of the real Haar MRA and the wavelet basis generated by $\psi^{(0)}$ is an analog of real Haar wavelet basis. But in contrast to the real setting, the *refinable function* ϕ generating our Haar MRA is *periodic* with the period 1 (see (4.11)), which *never holds* for real refinable functions. It will be shown bellow that due of this specific property of ϕ, there exist infinity many different orthonormal wavelet bases in the same Haar MRA (see Sec. 5).

Due to (2.3), (2.7), the function $\psi^{(0)}$ can be rewritten in the form $\psi^{(0)}(x)=\chi_2(2^{-1}x)\Omega(|x|_2)$ and the Haar wavelet basis is

$$\psi^{(0)}_{\gamma a}(x)=2^{-\gamma/2}\psi^{(0)}(2^{\gamma}x-a)$$
$$=2^{-\gamma/2}\chi_2\big(2^{-1}(2^{\gamma}x-a)\big)\Omega\big(|2^{\gamma}x-a|_2\big),\quad x\in\mathbb{Q}_2,\quad \gamma\in\mathbb{Z},\quad a\in I_2. \tag{4.15}$$

It is clear that

$$\int_{\mathbb{Q}_2}\psi^{(0)}_{\gamma a}(x)\,dx=0, \tag{4.16}$$

and, according to Lemma 3.1, $\psi^{(0)}_{\gamma a}(x)$ belongs to the Lyzorkin space $\Phi(\mathbb{Q}_2)$.

Remark 4.1. The Haar wavelet basis (4.15) coincides with Kozyrev's wavelet basis (1.2) for the case $p=2$. In present paper we restrict ourself by constructing the Haar wavelets only for $p=2$. Since Haar refinement equation (4.7) was presented for all p, a similar construction may be easily realized in the general case. Moreover, it is not difficult to see that Kozytev's wavelet function $\theta_j(x)$ from (1.2) can be expressed in terms of the *refinable function* $\phi(x)$ as

$$\theta_j(x)=\chi_p(p^{-1}jx)\Omega\big(|x|_p\big)=p^{-1/2}\sum_{r=0}^{p-1}h_r\phi\Big(\frac{1}{p}x-\frac{r}{p}\Big),\quad x\in\mathbb{Q}_p, \tag{4.17}$$

where $h_r = p^{1/2}e^{2\pi i\{\frac{jr}{p}\}_p}$, $r = 0, 1, \dots, p-1$, $j = 1, 2, \dots, p-1$.

Remark 4.2. In view of periodicity (4.11) of the refinable function ϕ, one can use shifts $\psi^{(0)}(\cdot + a)$, $a \in I_2$, instead of shifts $\psi^{(0)}(\cdot - a)$, $a \in I_2$.

Now we show that there is another function $\psi^{(1)}(x)$ whose shifts form an orthonormal basis in W_0. Indeed, taking into account (4.11), we have

$$\psi^{(1)}(x) = \frac{1}{\sqrt{2}}\Big(\phi\Big(\frac{x}{2}\Big) - \phi\Big(\frac{x}{2} - \frac{1}{2}\Big) - \phi\Big(\frac{x}{2} + \frac{1}{2^2}\Big) + \phi\Big(\frac{x}{2} - \frac{1}{2^2}\Big)\Big)$$

$$= \frac{1}{\sqrt{2}}\Big(\phi\Big(\frac{x}{2}\Big) - \phi\Big(\frac{x}{2} - \frac{1}{2}\Big) - \phi\Big(\frac{x}{2} - \frac{1}{2^2} - \frac{1}{2}\Big) + \phi\Big(\frac{x}{2} - \frac{1}{2^2}\Big)\Big) \tag{4.18}$$

an its shifts

$$\psi^{(1)}\Big(x + \frac{1}{2}\Big) = \frac{1}{\sqrt{2}}\Big(\phi\Big(\frac{x}{2} + \frac{1}{2^2}\Big) - \phi\Big(\frac{x}{2} - \frac{1}{2^2}\Big) - \phi\Big(\frac{x}{2} + \frac{1}{2}\Big) + \phi\Big(\frac{x}{2}\Big)\Big)$$

$$= \frac{1}{\sqrt{2}}\Big(\phi\Big(\frac{x}{2} - \frac{1}{2^2} - \frac{1}{2}\Big) - \phi\Big(\frac{x}{2} - \frac{1}{2^2}\Big) - \phi\Big(\frac{x}{2} - \frac{1}{2}\Big) + \phi\Big(\frac{x}{2}\Big)\Big), \tag{4.19}$$

$$\psi^{(1)}\big(x - a\big)$$

$$= \frac{1}{\sqrt{2}}\Big(\phi\Big(\frac{x}{2} - \frac{a}{2}\Big) - \phi\Big(\frac{x}{2} - \frac{a}{2} - \frac{1}{2}\Big) - \phi\Big(\frac{x}{2} - \frac{a}{2} + \frac{1}{2^2}\Big) + \phi\Big(\frac{x}{2} - \frac{a}{2} - \frac{1}{2^2}\Big)\Big).$$

$$= \frac{1}{\sqrt{2}}\Big(\phi\Big(\frac{x}{2} - \frac{a}{2}\Big) - \phi\Big(\frac{x}{2} - \frac{a}{2} - \frac{1}{2}\Big) - \phi\Big(\frac{x}{2} - \frac{a}{2} - \frac{1}{2^2} - \frac{1}{2}\Big) + \phi\Big(\frac{x}{2} - \frac{a}{2} - \frac{1}{2^2}\Big)\Big). \tag{4.20}$$

Since the system of functions $\{\phi(2^{-1}x - a) : a \in I_2\}$ is orthonormal, in view of (4.11), formulas (4.18)–(4.20) imply that the function $\psi^{(1)}(x)$ and the function $\psi^{(1)}(x - a)$ are orthonormal, whenever $a \in I_2$, $a \ne 0, \frac{1}{2}$. Here we take into account that all shifts (up to $\mod 1$) of refinable function in (4.18), (4.20) are distinct.

Similarly, by (4.18), (4.19), we have

$$\big(\psi^{(1)}(x), \psi^{(1)}(x + 2^{-1})\big) = \int_{\mathbb{Q}_2} \psi^{(1)}(x)\psi^{(1)}(x + 2^{-1})\,dx$$

$$= 2^{-1}\int_{\mathbb{Q}_2} \Big\{\phi^2\Big(\frac{x}{2}\Big) + \phi^2\Big(\frac{x}{2} - \frac{1}{2}\Big)$$

$$-\phi^2\Big(\frac{x}{2} - \frac{1}{2^2} - \frac{1}{2}\Big) - \phi^2\Big(\frac{x}{2} - \frac{1}{2^2}\Big)\Big\}\,dx = 0.$$

and

$$\big(\psi^{(1)}(x), \psi^{(1)}(x)\big) = 2^{-1}\int_{\mathbb{Q}_2} \Big(\phi^2\Big(\frac{x}{2}\Big) + \phi^2\Big(\frac{x}{2} - \frac{1}{2}\Big)$$

$$+\phi^2\Big(\frac{x}{2} - \frac{1}{2^2} - \frac{1}{2}\Big) + \phi^2\Big(\frac{x}{2} - \frac{1}{2^2}\Big)\Big)\,dx = 1.$$

Thus all shifts of $\psi^{(1)}$ are orthonormal.

It is clear that the functions (4.18) and (4.19) can be rewritten in the form

$$\psi^{(1)}(x) = \frac{1}{\sqrt{2}}\Big(\psi^{(0)}\big(x\big) - \psi^{(0)}\Big(x + \frac{1}{2}\Big)\Big), \tag{4.21}$$

$$\psi^{(1)}\Big(x + \frac{1}{2}\Big) = \frac{1}{\sqrt{2}}\Big(\psi^{(0)}\big(x\big) + \psi^{(0)}\Big(x + \frac{1}{2}\Big)\Big).$$

It follows that

$$\psi^{(0)}(x) = \frac{1}{\sqrt{2}}\Big(\psi^{(1)}\big(x\big) + \psi^{(1)}\Big(x + \frac{1}{2}\Big)\Big).$$

Since the system $\psi^{(0)}(\cdot - a)$, $a \in I_2$, forms an orthonormal basis for W_0, the system $\psi^{(1)}(\cdot - a)$, $a \in I_2$, is another orthonormal basis for W_0.

So, we showed that a wavelet basis generated by the Haar MRA is not unique.

5. Description of 2-adic Haar bases

5.1. **Complex wavelets.** Using the fact that all dilatations and shifts ($x \to 2^{\gamma}x + a$, $a \in I_2$) of the Haar wavelet function $\psi^{(0)}$ form a orthonormal basis in $\mathcal{L}^2(\mathbb{Q}_2)$, we show that there exist infinitely many wavelet functions $\psi^{(s)}$, $s \in \mathbb{N}$ in W_0.

In what follows, we shall write the 2-adic number $a = 2^{-s}\big(a_0 + a_1 2 + \cdots + a_{s-1}2^{s-1}\big) \in I_2$, $a_j = 0, 1$, $j = 0, 1, \dots, s-1$ briefly as a rational number $a = \frac{m}{2^s}$, where $m = a_0 + a_1 2 + \cdots + a_{s-1}2^{s-1}$.

Since the characteristic function of the unit disc $\phi(x) = \Delta_0(x) = \Omega\big(|x|_2\big)$ is *periodic* with the period $\xi \in S_0$, the wavelet function $\psi^0(x)$ has the following evident and important property:

$$\psi^{(0)}(x + \xi) = -\psi^{(0)}(x), \qquad \xi \in S_0. \tag{5.1}$$

Here $\xi = 1 + \xi_1 2 + \xi_2 2^2 + \cdots$, where $\xi_j = 0, 1$; $j \in \mathbb{N}$.

Before we prove a general result, we consider the simplest particular case. Consider the function

$$\psi^{(1)}(x) = \alpha_0\psi^{(0)}\big(x\big) + \alpha_1\psi^{(0)}\Big(x + \frac{1}{2}\Big), \quad \alpha_0, \alpha_1 \in \mathbb{C}, \tag{5.2}$$

and solve the problem when all shifts of this function generates an orthonormal basis $\psi^{(1)}(x + a)$, $a \in I_2$ in W_0.

Taking into account orthonormality of the system $\psi^{(0)}(x + a)$, $a \in I_2$ and relation (5.1), we can see that the function $\psi^{(1)}(x)$ and the functions $\psi^{(1)}(x+a)$ are orthonormal for all $a \in I_2$, $a \neq 0, \frac{1}{2}$. Thus, in view of (5.1), the system of functions $\psi^{(1)}(x + a)$, $a \in I_2$ is orthonormal if and only if the system of functions (5.2) and

$$\psi^{(1)}\Big(x + \frac{1}{2}\Big) = -\alpha_1\psi^{(0)}\big(x\big) + \alpha_0\psi^{(0)}\Big(x + \frac{1}{2}\Big) \tag{5.3}$$

is orthonormal. Hence, we have $|\alpha_0|^2+|\alpha_1|^2=1$. In other words, the matrix

$$D=\begin{pmatrix}\alpha_0 & \alpha_1\\ -\alpha_1 & \alpha_0\end{pmatrix}$$

is unitary. Thus, the function (5.2), where $|\alpha_0|^2+|\alpha_1|^2=1$ is the wavelet function. It is clear that the wavelet function (4.21) is a particular case of the wavelet function (5.2).

Consequently, all dilatations and shifts of $\psi^{(1)}(x)$ form 2-adic orthonormal wavelet basis in $\mathcal{L}^2(\mathbb{Q}_2)$.

Now we will prove a general theorem.

Theorem 5.1. *Let $s=1,2,\dots$. The function*

$$\psi^{(s)}(x)=\sum_{k=0}^{2^s-1}\alpha_k\psi^{(0)}\Big(x+\frac{k}{2^s}\Big), \tag{5.4}$$

is the wavelet function (whose dilatations and shifts form 2-adic orthonormal wavelet basis in $\mathcal{L}^2(\mathbb{Q}_2)$) if and only if

$$\alpha_k=2^{-s}(-1)^k\sum_{r=0}^{2^s-1}\gamma_r e^{-i\pi\frac{2r+1}{2^s}k}, \quad k=0,1,2,\dots,2^s-1, \tag{5.5}$$

$\gamma_k\in\mathbb{C}$, $|\gamma_k|=1$.

Proof. Suppose that $\psi^{(s)}(x)$, $s\ge 1$ is given by formula (5.4). Since the system $\psi^{(0)}(\cdot+a)$, $a\in I_2$ is orthonormal (see Subsec. 4.3) and in view of relation (5.1), it is easy to see that $\psi^{(s)}$ and $\psi^{(s)}(\cdot+a)$ are orthonormal for any $a\in I_2$, $a\ne\frac{k}{2^s}$, $k=0,1,\dots 2^s-1$. Thus the system of functions $\psi^{(s)}(x+a)$, $a\in I_2$ is orthonormal if and only if the system of functions, consisting of the function (5.4) and its shifts, i.e.,

$$\begin{aligned}\psi^{(s)}\Big(x+\frac{r}{2^s}\Big)&=-\alpha_{2^s-r}\psi^{(0)}(x)-\alpha_{2^s-r+1}\psi^{(0)}\Big(x+\frac{1}{2^s}\Big)-\dots-\alpha_{2^s-1}\psi^{(0)}\Big(x+\frac{r-1}{2^s}\Big)\\ &\quad+\alpha_0\psi^{(0)}\Big(x+\frac{r}{2^s}\Big)+\dots+\alpha_{2^s-r-1}\psi^{(0)}\Big(x+\frac{2^s-1}{2^s}\Big),\end{aligned} \tag{5.6}$$

$r=0,1,\dots,2^s-1$ is orthonormal.

Set $\Xi^{(0)}=\{\psi^{(0)}(\cdot+\frac{k}{2^s}):k=0,1,\dots,2^s-1\}^T$, $\Xi^{(s)}=\{\psi^{(s)}(\cdot+\frac{k}{2^s}):k=0,1,\dots,2^s-1\}^T$. In view of (5.4), (5.6), $\Xi^{(s)}=D\Xi^{(0)}$, where

$$D=\begin{pmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_{2^s-2} & \alpha_{2^s-1}\\ -\alpha_{2^s-1} & \alpha_0 & \alpha_1 & \dots & \alpha_{2^s-3} & \alpha_{2^s-2}\\ -\alpha_{2^s-2} & -\alpha_{2^s-1} & \alpha_0 & \dots & \alpha_{2^s-4} & \alpha_{2^s-3}\\ \dots & \dots & \dots & \dots & \dots & \dots\\ -\alpha_2 & -\alpha_3 & -\alpha_4 & \dots & \alpha_0 & \alpha_1\\ -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{2^s-1} & \alpha_0\end{pmatrix}. \tag{5.7}$$

Thus the system $\Xi^{(s)}$ is orthonormal if and only if the matrix D is unitary.

Let $u = (\alpha_0, \alpha_1, \dots, \alpha_{2^s-1})^T$ be a vector and

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -1 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

be a $2^s \times 2^s$ matrix. It is easy to see that

$$A^r u = (-\alpha_{2^s-r}, -\alpha_{2^s-r+1}, \dots, -\alpha_{2^s-1}, \alpha_0, \alpha_1, \dots, \alpha_{2^s-r-1})^T,$$

$r = 1, 2, \dots, 2^s - 1$. Thus $D = \big(u, Au, \dots, A^{2^s-1}u\big)^T$. It is significant that $A^{2^s}u = -u$. Consequently, in order to describe all matrixes D (or in other words, all vectors u), we should find all vectors $u = (\alpha_0, \alpha_1, \dots, \alpha_{2^s-1})^T$ such that the system $\{A^r u : r = 0, 1, 2, \dots, 2^s - 1\}$ is orthonormal.

In view of the fact that the system $\psi^{(0)}(x+a)$, $a \in I_2$ forms an orthonormal basis in W_0, it is easy to see that the vector $u_0 = (1, 0, \dots, 0, 0)^T$ is one of mentioned above vectors u. That is the system composed of vectors u_0 and $A^r u_0 = (\delta_{0\,r}, \delta_{1\,r}, \dots, \delta_{2^s-2\,r}, \delta_{2^s-1\,r})^T$, $r = 1, 2, \dots, 2^s-1$, is orthonormal, where $\delta_{i\,r}$ is the Kronecker symbol.

Let us prove that the vector $u = (\alpha_0, \alpha_1, \dots, \alpha_{2^s-1})^T$ already mentioned above such that $A^r u$, $r = 0, 1, 2, \dots, 2^s - 1$ is orthonormal, can be expressed by the formula $u = Bu_0$ if and only if B is a unitary matrix such that $AB = BA$. Indeed, let $u = Bu_0$, where B is a unitary matrix such that $AB = BA$. Then $A^r u = BA^r u_0$, $r = 0, 1, 2, \dots, 2^s - 1$. Since the system $A^r u_0$, $r = 0, 1, 2, \dots, 2^s - 1$ is orthonormal and the matrix B is unitary, the vectors $A^r u$, $r = 0, 1, 2, \dots, 2^s - 1$ are orthonormal. Conversely, if the system $A^r u$, $r = 0, 1, 2, \dots, 2^s-1$ is orthonormal, taking into account that the system $A^r u_0$, $r = 0, 1, 2, \dots, 2^s - 1$ is orthonormal, we conclude that there exists a unitary matrix B such that $A^r u = B(A^r u_0)$, $r = 0, 1, 2, \dots, 2^s - 1$. Since $A^{2^s}u = -u$, $A^{2^s}u_0 = -u_0$, we have an additional relation $A^{2^s}u = BA^{2^s}u_0$. It follows from the above relations that $(AB - BA)(A^r u_0) = 0$, $r = 0, 1, 2, \dots, 2^s - 1$. Since the vectors $A^r u_0$, $r = 0, 1, 2, \dots, 2^s - 1$ form a basis in the 2^s-dimensional space, we conclude that $AB = BA$.

Thus we have $D = \big(Bu_0, BAu_0, \dots, BA^{2^s-1}u_0\big)^T$.

It is clear that the eigenvalues of A and the corresponding normalized eigenvectors are

$$\lambda_r = -e^{i\pi\frac{2r+1}{2^s}}, \tag{5.8}$$

and $v_r = \big((v_r)_1, \dots, (v_r)_{2^s}\big)^T$, respectively, where

$$(v_r)_l = 2^{-s/2}(-1)^l e^{-i\pi\frac{2r+1}{2^s}l}, \quad l = 0, 1, 2, \dots, 2^s - 1, \tag{5.9}$$

$r = 0, 1, 2, \dots, 2^s - 1$. As is well known, the matrix A can be represented as $A = C\widetilde{A}C^{-1}$, where

$$\widetilde{A} = \begin{pmatrix} \lambda_0 & 0 & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{2^s-1} \end{pmatrix}$$

is a diagonal matrix, $C = \big(v_0, v_1, \dots, v_{2^s-1}\big)$. Since C is a unitary matrix, the matrix $B = C\widetilde{B}C^{-1}$ is unitary if and only if $\widetilde{B}$ is unitary. On the other hand, $AB = BA$ if and only if $\widetilde{A}\widetilde{B} = \widetilde{B}\widetilde{A}$. Moreover, since according to (5.8) $\lambda_k \ne \lambda_l$, whenever $k \ne l$, all unitary matrix $\widetilde{B}$ such that $\widetilde{A}\widetilde{B} = \widetilde{B}\widetilde{A}$, are given by

$$\widetilde{B} = \begin{pmatrix} \gamma_0 & 0 & \dots & 0 \\ 0 & \gamma_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_{2^s-1} \end{pmatrix},$$

where $\gamma_k \in \mathbb{C}$, $|\gamma_k| = 1$. Hence, all unitary matrix B such that $AB = BA$, are given by $B = C\widetilde{B}C^{-1}$, where $\widetilde{B}$ is the above diagonal matrix.

By using formula (5.9), one can calculate

$$\alpha_k = (Bu_0)_k = (C\widetilde{B}C^{-1}u_0)_k = \sum_{r=0}^{2^s-1} \gamma_r (v_r)_k (\overline{v}_r)_0$$

$$= 2^{-s}(-1)^k \sum_{r=0}^{2^s-1} \gamma_r e^{-i\pi\frac{2r+1}{2^s}k}, \quad k = 0, 1, 2, \dots, 2^s - 1,$$

where $\gamma_k \in \mathbb{C}$, $|\gamma_k| = 1$. Thus (5.5) holds.

Taking into account that $\Xi^{(0)} = D^{-1}\Xi^{(s)}$, we conclude that if we define $\psi^{(s)}(x)$ by formula (5.4), where α_k is given by (5.5), $k = 0, 1, 2, \dots, 2^s - 1$, then the system of functions $\{\psi^{(s)}(\cdot - a) : a \in a \in I_2\}$ is orthonormal and forms the orthonormal basis in W_0.

Consequently, all dilatations and shifts of the function (5.4) form 2-adic orthonormal wavelet basis in $\mathcal{L}^2(\mathbb{Q}_2)$. □

It is clear that $\int_{\mathbb{Q}_2} \psi^{(s)}_{\gamma a}(x)\,dx = 0$, and in view of Lemma 3.1, $\psi^{(s)}_{\gamma a}(x)$ belongs to the Lizorkin space $\in \Phi(\mathbb{Q}_2^n)$.

5.2. **Real wavelets.** Using formulas (5.5), one can extract all *real* wavelet functions (5.4).

Let $s = 1$. According to (5.2), (5.3),

$$\psi^{(1)}(x) = \cos\theta\, \psi^{(0)}\big(x\big) + \sin\theta\, \psi^{(0)}\Big(x + \frac{1}{2}\Big) \tag{5.10}$$

is the *real* wavelet function.

Let $s=2$. Set $\gamma_r=e^{i\theta_r}$, $r=0,1,2,\dots,2^s-1$. Then (5.5) imply that the wavelet function $\psi^{(1)}(x)$ is real if and only if

$$\begin{array}{rcl} \sin\theta_1+\sin\theta_2+\sin\theta_3+\sin\theta_4 &=& 0,\\ \cos\theta_1-\cos\theta_2+\cos\theta_3-\cos\theta_4 &=& 0,\\ \sin\theta_1-\sin\theta_2-\sin\theta_3+\sin\theta_4 &=& \\ \cos\theta_1+\cos\theta_2-\cos\theta_3-\cos\theta_4, && \\ \sin\theta_1-\sin\theta_2-\sin\theta_3+\sin\theta_4 &=& \\ -(\cos\theta_1+\cos\theta_2-\cos\theta_3-\cos\theta_4). && \end{array}$$

The last relations are equivalent to the system

$$\begin{array}{rclcrcl} \sin\theta_1 &=& -\sin\theta_4, & \quad & \cos\theta_1 &=& \cos\theta_4,\\ \sin\theta_2 &=& -\sin\theta_3, & & \cos\theta_2 &=& \cos\theta_3. \end{array}$$

Thus for $s=2$ the real wavelet functions (5.4) is represented as

$$\begin{aligned} \psi^{(1)}(x) &= \frac{1}{2}(\cos\theta_1+\cos\theta_2)\psi^{(0)}\big(x\big)\\ &\quad+\frac{1}{2\sqrt{2}}(\cos\theta_1-\cos\theta_2+\sin\theta_1+\sin\theta_2)\psi^{(0)}\Big(x+\frac{1}{2^2}\Big)\\ &\quad+\frac{1}{2}(\sin\theta_1-\sin\theta_2)\psi^{(0)}\Big(x+\frac{1}{2}\Big)\\ &\quad+\frac{1}{2\sqrt{2}}(\cos\theta_1-\cos\theta_2-\sin\theta_1-\sin\theta_2)\psi^{(0)}\Big(x+\frac{1}{2^2}+\frac{1}{2}\Big). \end{aligned} \tag{5.11}$$

In particular, for the special cases $\theta_1=\theta_2=\theta$, $\theta_1=-\theta_2=\theta$, $\theta_1=\theta_2+\frac{\pi}{2}=\theta$, we obtain one-parameter families of the real wavelet functions

$$\begin{array}{rcl} \psi^{(1)}(x) &=& \cos\theta\psi^{(0)}\big(x\big)+\sin\theta\psi^{(0)}\Big(x+\frac{1}{2}\Big),\\ \psi^{(1)}(x) &=& \cos\theta\psi^{(0)}\big(x\big)+\frac{1}{\sqrt{2}}\sin\theta\psi^{(0)}\Big(x+\frac{1}{2^2}\Big)\\ && \qquad -\frac{1}{\sqrt{2}}\sin\theta\psi^{(0)}\Big(x+\frac{1}{2^2}+\frac{1}{2}\Big),\\ \psi^{(1)}(x) &=& \frac{1}{2}(\cos\theta-\sin\theta)\psi^{(0)}\big(x\big)+\frac{1}{2\sqrt{2}}(\cos\theta+\sin\theta)\psi^{(0)}\Big(x+\frac{1}{2^2}\Big)\\ && \qquad -\frac{1}{2}(\cos\theta-\sin\theta)\psi^{(0)}\Big(x+\frac{1}{2}\Big), \end{array} \tag{5.12}$$

respectively.

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